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Hot electrons under quantization conditions: II. The Boltzmann equation

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Abstract. A general analysis of the Boltzmann equation is undertaken for the non-degenerate electrons in low-dimensional system interacting with bulk acoustic phonons. We derive expressions for symmetric and antisymmetric collision operators for electrons confined by an arbitrary quantizing potential. These expressions are universal and can be applied to three-, two- or one-dimensional electron gases. As was shown in our previous paper (part I) the kinetics of the low-dimensional electron system is qualitatively different at low and high lattice temperatures due to the kinematic peculiarities of the electron–acoustic-phonon interaction. Here we present the Boltzmann equation to describe the non-equilibrium electron distribution in low-dimensional electron systems under different external conditions which are of physical interest and which will be investigated in the following paper, part III.

1. Introduction

In the previous paper [1] (hereafter referred to as I) the basic kinematics of the interaction between electrons completely confined in a quantum wire (QWI) and bulk acoustic phonons via the deformation potential was described in depth, as an essential pre-requisite for a discussion of hot-electron transport. The next step is the examination of the Boltzmann equation and the form of its solutions, which is the topic of this paper. In section 2 we describe the Boltzmann equation in terms of symmetric and antisymmetric distribution functions. The properties of the antisymmetric collision operator and the conditions under which a momentum relaxation time can be defined are discussed in section 3. Section 4 deals with the general form of the symmetric collision operator for quasi-elastic processes, which is extended to inelastic processes in section 5. The results are summarized and discussed in section 6.

2. General expressions

We will assume in the following that we are dealing with a non-degenerate, spatially homogeneous 1D electron gas. For the stationary case the Boltzmann equation has the form

$$-\frac{eE_x}{\hbar}\frac{\mathrm{d}F_\nu(\kappa_x)}{\mathrm{d}\kappa_x} = \hat{I}F_\nu(\kappa_x) \tag{2.1}$$

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where $F_\nu(\kappa_x)$ is the electron distribution function in sub-band ν, E_x is an applied uniform electric field and all other notations here and below are the same as in the previous paper I. The collision operator $\hat{I}F_\nu(\kappa_x)$ is given by

$$\hat{I}F_\nu(\kappa_x) = \sum_{\nu'}\sum_{\kappa_x'}\sum_{\boldsymbol{q}}[W_{\nu'\nu}(\kappa_x', \kappa_x, \boldsymbol{q})F_{\nu'}(\kappa_x') - W_{\nu\nu'}(\kappa_x, \kappa_x', \boldsymbol{q})F_\nu(\kappa_x)] \tag{2.2}$$

where

$$W_{\nu\nu'}(\kappa_x, \kappa_x', \boldsymbol{q}) = W^+_{\nu\nu'}(\kappa_x, \kappa_x', \boldsymbol{q}) + W^-_{\nu\nu'}(\kappa_x, \kappa_x', \boldsymbol{q})$$
$$W^\pm_{\nu\nu'}(\kappa_x, \kappa_x', \boldsymbol{q}) = \frac{2\pi}{\hbar}|M^\pm_{\nu\nu'}(\kappa_x, \kappa_x', \boldsymbol{q})|^2(N_{\boldsymbol{q}} + \tfrac{1}{2} \pm \tfrac{1}{2})\delta(\varepsilon_{\nu'}(\kappa_x') - \varepsilon_\nu(\kappa_x) \pm \hbar\omega_{\boldsymbol{q}}). \tag{2.3}$$

On substituting $W^\pm_{\nu\nu'}(\kappa_x, \kappa_x', \boldsymbol{q})$ from equation (2.3) into equation (2.2) we obtain, after summation over κ_x' with the use of the Kronecker delta functions $\delta_{\kappa_x', \kappa_x \mp q_x}$ the following expression

$$\begin{aligned}\hat{I}F_\nu(\kappa_x) = \sum_{\nu'}\sum_{\boldsymbol{q}} w(q)G^2_{\nu\nu'}(\boldsymbol{q}_\perp)\{&[F_{\nu'}(\kappa_x + q_x)(N_{\boldsymbol{q}} + 1) - F_\nu(\kappa_x)N_{\boldsymbol{q}}] \\ &\times\delta[\varepsilon_{\nu'}(\kappa_x + q_x) - \varepsilon_\nu(\kappa_x) - \hbar sq] + [F_{\nu'}(\kappa_x - q_x)N_{\boldsymbol{q}} - F_\nu(\kappa_x)(N_{\boldsymbol{q}} + 1)] \\ &\times\delta[\varepsilon_{\nu'}(\kappa_x - q_x) - \varepsilon_\nu(\kappa_x) + \hbar sq]\}\end{aligned} \tag{2.4}$$

where

$$w(q) = w_0 q \qquad w_0 = \frac{\pi\,\Xi_a^2}{\rho V_0 s}. \tag{2.5}$$

For the solution of equation (2.1) let us represent the distribution function $F_\nu(\kappa_x)$ as a sum of symmetric (even) and antisymmetric (odd) parts

$$F_\nu(\kappa_x) = F^+_\nu(\kappa_x) + F^-_\nu(\kappa_x) \tag{2.6}$$

where

$$F^+_\nu(\kappa_x) = \tfrac{1}{2}[F_\nu(\kappa_x) + F_\nu(-\kappa_x)] \tag{2.7}$$
$$F^-_\nu(\kappa_x) = \tfrac{1}{2}[F_\nu(\kappa_x) - F_\nu(-\kappa_x)]. \tag{2.8}$$

Because the function $F^+_\nu(\kappa_x)$ is an even function of κ_x and is related to the state (ν, κ_x) we can consider that

$$F^+_\nu(\kappa_x) \equiv F_{0\nu}(\varepsilon_\nu(\kappa_x)). \tag{2.9}$$

The antisymmetric function $F^-_\nu(\kappa_x)$ has to be proportional to κ_x, $F^-_\nu(\kappa_x) \propto \kappa_x$. This is a unique feature of the 1D electron gas (it is true also for longitudinal (along a magnetic field) transport of electrons in a quantizing magnetic field). In 3D or 2D electron gases the representation of the distribution function in the form of equation (2.6) is equivalent to an expansion in a series of Legendre polynomials with argument equal to the cosine of the angle between $\boldsymbol{\kappa}$ and $\boldsymbol{E}$, and $F^\pm(\kappa)$ contains the sum of all even or odd harmonics, respectively. As a rule, it is quite a good approximation in these cases to truncate these series beyond the first terms for even and odd functions (for details see, for example, [2]), namely $F^+(\boldsymbol{\kappa}) \propto (\boldsymbol{\kappa}\cdot\boldsymbol{E})^0$ and $F^-(\boldsymbol{\kappa}) \propto (\boldsymbol{\kappa}\cdot\boldsymbol{E})$, and all other harmonics are ignored. For a 1D electron gas only the linear harmonic for the odd function $F^-_\nu(\kappa_x)$, which is proportional to κ_x, exists. All other odd harmonics can be expressed through the first one, because $\kappa_x^{2n+1} \simeq \varepsilon^n_\parallel(\kappa_x)\kappa_x$ for any integer n. Hence, we can present the antisymmetric function $F^-_\nu(\kappa_x)$ in the form

$$F^-_\nu(\kappa_x) = \kappa_x f_\nu(\varepsilon_\nu(\kappa_x)) \tag{2.10}$$

without any restriction of generality. The same generality is true for the symmetric function $F_\nu^+(\kappa_x)$ in equation (2.9).

Making use of equations (2.6), (2.9) and (2.10), we obtain the following system of kinetic equations:

$$-\frac{eE_x}{\overline{h}}\frac{\mathrm{d}F_{0\nu}(\varepsilon_\nu(\kappa_x))}{\mathrm{d}\kappa_x} = \hat{I}F_\nu^-(\kappa_x) \tag{2.11}$$

$$-\frac{eE_x}{\overline{h}}\frac{\mathrm{d}F_\nu^-(\kappa_x)}{\mathrm{d}\kappa_x} = \hat{I}F_{0\nu}(\varepsilon_\nu(\kappa_x)) \tag{2.12}$$

where

$$\begin{aligned}\hat{I}F_\nu^-(\kappa_x) = -F_\nu(\kappa_x)\sum_{\nu'}\sum_{\boldsymbol{q}} w(q)G^2_{\nu\nu'}(\boldsymbol{q}_\perp)\Bigg\{\Bigg[N_{\boldsymbol{q}} - \left(1+\frac{q_x}{\kappa_x}\right)\frac{f_{\nu'}[\varepsilon_{\nu'}(\kappa_x+q_x)]}{f_\nu(\varepsilon_\nu(\kappa_x))}(N_{\boldsymbol{q}}+1)\Bigg]\\ \times\delta[\varepsilon_{\nu'}(\kappa_x+q_x)-\varepsilon_\nu(\kappa_x)-\overline{h}sq]\\ +\Bigg[(N_{\boldsymbol{q}}+1) - \left(1-\frac{q_x}{\kappa_x}\right)\frac{f_{\nu'}[\varepsilon_{\nu'}(\kappa_x-q_x)]}{f_\nu(\varepsilon_\nu(\kappa_x))}N_{\boldsymbol{q}}\Bigg]\\ \times\delta[\varepsilon_{\nu'}(\kappa_x-q_x)-\varepsilon_\nu(\kappa_x)+\overline{h}sq]\Bigg\}\end{aligned} \tag{2.13}$$

$$\begin{aligned}\hat{I}F_{0\nu}(\varepsilon_\nu(\kappa_x)) = \sum_{\nu'}\sum_{\boldsymbol{q}} w(q)G^2_{\nu\nu'}(\boldsymbol{q}_\perp)\{\{F_{0\nu'}[\varepsilon_\nu(\kappa_x+q)](N_{\boldsymbol{q}}+1) - F_{0\nu}(\varepsilon_\nu(\kappa_x))N_{\boldsymbol{q}}\}\\ \times\delta[\varepsilon_{\nu'}(\kappa_x+q_x)-\varepsilon_\nu(\kappa_x)-\overline{h}sq]\\ +\{F_{0\nu'}[\varepsilon_{\nu'}(\kappa_x-q_x)]N_{\boldsymbol{q}} - F_{0\nu}(\varepsilon_\nu(\kappa_x))(N_{\boldsymbol{q}}+1)\}\\ \times\delta[\varepsilon_{\nu'}(\kappa_x-q_x)-\varepsilon_\nu(\kappa_x)+\overline{h}sq]\}.\end{aligned} \tag{2.14}$$

Some general conclusions can be drawn concerning the antisymmetric collision operator in equation (2.13).

3. Properties of the antisymmetric collision operator

If the electron kinetic energy $\varepsilon_\parallel(\kappa_x)$ in sub-band ν is within the range

$$\varepsilon_\parallel(\kappa_x) = \varepsilon_\nu(\kappa_x) - W_\nu > (8m^*s^2W_0)^{1/2} \tag{3.1}$$

then, in accordance with paper I (equations (5.3) and (5.26) in I), the electron–acoustic-phonon interaction is quasi-elastic both for intra-sub-band and for inter-sub-band scattering. To calculate the relaxation rate $\hat{I}F_\nu^-(\kappa_x)$ for this case we can neglect the phonon energy $\overline{h}\omega_{\boldsymbol{q}}$ in the arguments of the delta-functions in equation (2.13) (the elastic approximation) and we obtain

$$\hat{I}F_\nu^-(\kappa_x) = -\sum_{\nu'}\left(\frac{F_\nu^-(\kappa_x)}{\tau^{(+)}_{\nu\nu'}(\kappa_x)} - \frac{F_{\nu'}^-(\kappa_x)}{\tau^{(-)}_{\nu'\nu}(\kappa_x)}\right) \tag{3.2}$$

where

$$\frac{1}{\tau^{(+)}_{\nu\nu'}(\kappa_x)} = \sum_{\boldsymbol{q}} w(q)G^2_{\nu\nu'}(\boldsymbol{q}_\perp)(2N_{\boldsymbol{q}}+1)\left(1+\frac{q_x}{2\kappa_x}\mp\frac{q_x}{2\kappa_x}\right)\delta[\varepsilon_{\nu'}(\kappa_x+q_x)-\varepsilon_\nu(\kappa_x)]. \tag{3.3}$$

Here the upper sign corresponds to scattering out of a κ_x state in the sub-band ν and the lower sign to scattering into the κ_x state due both to emission and to absorption of acoustic phonons. Equations (3.2) and (3.3) show us that, in a 1D QWI, even in the elastic approximation, we have a set of relaxation times $\tau^{(\pm)}_{\nu\nu'}(\kappa_x)$ to describe the relaxation of the antisymmetric function $F_\nu^-(\kappa_x)$ in sub-band ν rather than one momentum relaxation time.

However, only one relaxation time exists if the 1D electrons occupy only the first sub-band ($\nu = \nu' = 1$), or in the case in which the product $[w(q)(2N_{\boldsymbol{q}} + 1)]$ in equation (3.3) does not depend on the q_x component of the phonon wavevector $\boldsymbol{q}$. Situations in which this product does not depend on q_x will be analysed later. For both these cases equation (3.2) is transformed into the form

$$\hat{I}F_\nu^-(\kappa_x) = -F_\nu^-(\kappa_x)\frac{1}{\tau_\nu(\kappa_x)} \tag{3.4}$$

where the momentum relaxation time $\tau_\nu(\kappa_x)$ is given by

$$\frac{1}{\tau_1(\kappa_x)} = \sum_{\boldsymbol{q}} w(q)G_{11}^2(\boldsymbol{q}_\perp)(2N_{\boldsymbol{q}} + 1)\left[1 - \left(1 + \frac{q_x}{\kappa_x}\right)\right]\delta[\varepsilon_1(\kappa_x + q_x) - \varepsilon_1(\kappa_x)] \tag{3.5}$$

for the case of occupation only of the first sub-band ($\nu' = \nu = 1$) and

$$\frac{1}{\tau_\nu(\kappa_x)} = \sum_{\nu'}\sum_{\boldsymbol{q}} w(q)G_{\nu\nu'}^2(\boldsymbol{q}_\perp)(2N_{\boldsymbol{q}} + 1)\delta[\varepsilon_{\nu'}(\kappa_x + q_x) - \varepsilon_\nu(\kappa_x)] \tag{3.6}$$

for the case in which the product $w(q)(2N_{\boldsymbol{q}} + 1)$ does not depend on q_x. To derive equation (3.6) from equation (3.3) we have used the following relation:

$$\sum_{q_x}\left(1 \pm \frac{q_x}{\kappa_x}\right)\Phi_{evn}(\kappa_x \pm q_x)\delta[\varphi_{evn}(\kappa_x \pm q_x) - \varphi_0] = 0 \tag{3.7}$$

which is true for arbitrary even functions Φ_{evn} and φ_{evn}, and for φ_0 which is independent of q_x. As a consequence of equation (3.7), only scattering from a κ_x state contributes to the momentum relaxation time in equation (3.6) and the momentum relaxation time is equal to the scattering time. Of course, if the above-mentioned product does not depend on q_x as well in equation (3.5) then the second term in square brackets in equation (3.5) will be eliminated due to the property described by equation (3.7) and equation (3.5) will coincide with equation (3.6) for $\nu' = \nu = 1$.

If the electron kinetic energy in sub-band ν is within the range

$$\varepsilon_\parallel(\kappa_x) = \varepsilon_\nu(\kappa_x) - W_\nu < (8m^*s^2W_0)^{1/2} \tag{3.8}$$

then the electron–acoustic-phonon interaction is inelastic for intra-sub-band ($\nu' = \nu$) scattering and quasi-elastic for inter-sub-band ($\nu' \neq \nu$) scattering. Then, as was shown in paper I, we can use the relationship $q_\perp \gg |q_x|$ (see equation (5.5) in I) to calculate the term with $\nu' = \nu$ in the sum of equation (2.13); that is, we can put $q \simeq q_\perp$ for this term and we can use the elastic approximation to calculate terms with $\nu' \neq \nu$ in this equation, namely we can neglect $\hbar\omega_{\boldsymbol{q}}$ in the arguments of the delta-functions for these terms. Making use of these approximations we obtain for the antisymmetric collision operator

$$\hat{I}F_\nu^-(\kappa_x) = -\frac{F_\nu^-(\kappa_x)}{\tau_{0\nu}(\kappa_x)} - \sum_{\nu'\neq\nu}\left(\frac{F_\nu^-(\kappa_x)}{\tau_{\nu\nu'}^{(+)}(\kappa_x)} - \frac{F_\nu^-(\kappa_x)}{\tau_{\nu'\nu}^{(-)}(\kappa_x)}\right) \tag{3.9}$$

where

$$\frac{1}{\tau_{0\nu}(\kappa_x)} = \sum_{\boldsymbol{q}} w(q_\perp)G_{\nu\nu}^2(\boldsymbol{q}_\perp)\{N_{q_\perp}\delta[\varepsilon_\nu(\kappa_x + q_x) - \varepsilon_\nu(\kappa_x) - \hbar s q_\perp] + (N_{\boldsymbol{q}_\perp} + 1)$$
$$\times\delta[\varepsilon_\nu(\kappa_x - q_x) - \varepsilon_\nu(\kappa_x) + \hbar s q_\perp]\}. \tag{3.10}$$

We can call $\tau_{0\nu}(\kappa_x)$ the inelastic momentum relaxation time for intra-sub-band scattering. If T_0 is within the range $T_0 < \hbar s q_{\perp 0}$ and the electron energy is in the range $T_0 < \varepsilon_\parallel(\kappa_x) < (8m^*s^2W_0)^{1/2}$ then, in the first term in equation (3.10), which corresponds to the absorption processes, we can put $G_{\nu\nu}^2(\boldsymbol{q}_\perp) \simeq 1$ and neglect $\hbar s q_\perp$ in the delta-function argument.

To derive equation (3.10) we have used in equation (2.13) the property embodied in equation (3.7). Due to this only the scattering from the κ_x state contributes to the $\tau_{0\nu}(\kappa_x)$ and the inelastic momentum relaxation time is equal to the inelastic scattering time. Note that, for the discussed conditions, this is also true for a degenerate electron gas (due to equation (3.7)). This means that the expression for the momentum relaxation rate which was used in [3] is incorrect. As a result the authors [3] have obtained an unphysical result [4]: a negative transport lifetime of electrons in quantum wires due to their interaction with acoustic phonons.

In equation (3.9) we have, as in equation (3.2), a set of relaxation times. The unusual physical peculiarity of equation (3.9) is the combination of the contributions of inelastic electron–acoustic-phonon scattering (for intra-sub-band transitions) and elastic scattering (for inter-sub-band transitions) to describe the relaxation of the antisymmetric function $F_\nu^-(\kappa_x)$. Equation (3.9) can be transformed into the form as in equation (3.4), which describes a single relaxation time, only for the two cases discussed previously. If electrons populate only the first sub-band ($\nu = \nu' = 1$) then equation (3.9) coincides with equation (3.4) with the momentum relaxation time

$$\frac{1}{\tau_1(\kappa_x)} = \frac{1}{\tau_{01}(\kappa_x)} \tag{3.11}$$

where $\tau_{01}(\kappa_x)$ is given by equation (3.10).

If the product $w(q)(2N_q + 1)$ does not depend on q_x then $1/(\tau_{\nu\nu}^{(-1)}(\kappa_x)) = 0$, due to equation (3.7), and equation (3.9) is transformed into the same form as equation (3.4) with the momentum relaxation time which is given by

$$\frac{1}{\tau_\nu(\kappa_x)} = \frac{1}{\tau_{0\nu}(\kappa_x)} + \sum_{\nu' \neq \nu} \frac{1}{\tau_{\nu\nu'}^{(+)}(\kappa_x)}. \tag{3.12}$$

This general analysis shows us possible cases in which the antisymmetric collision operator $\hat{I}F_\nu^-(\kappa_x)$ can be transformed from the integral form in equation (2.13) into the *algebraic* form in equations (3.2) and (3.9), whereby we can introduce the momentum relaxation time approximation as in equation (3.4). Only in these cases is it possible to obtain analytical solutions of the Boltzmann equations (2.11) and (2.12). However, the realization of this possibility depends as well on the form of the symmetric collision operator $\hat{I}F_{0\nu}(\varepsilon_\nu(\kappa_x))$ from equation (2.14). There are three mathematical reasons which make the expression for $\hat{I}F_{0\nu}(\varepsilon_\nu(\kappa_x))$ for a 1D electron gas much more complicated than that for a 3D electron gas. There is first the summation over ν', then the presence of the form factor $G_{\nu\nu'}^2(\boldsymbol{q}_\perp)$, which depends on the transverse component $\boldsymbol{q}_\perp$ rather than on the modulus of q, and, finally, the delta-function arguments in equation (2.14), which contain separately the modulus q and its longitudinal component q_x.

4. The form of the symmetric collision operator for quasi-elastic scattering

As was shown in paper I, the character of the electron–acoustic-phonon interaction depends on the values of the electron kinetic energy as defined in equations (3.1) and (3.8). Depending on whether the electron kinetic energy in sub-band ν is within the range defined by equation (3.1) or within the range defined by equation (3.8), the symmetric collision operator in equation (2.14) transforms differently.

To derive these forms let us perform an averaging of the symmetric collision operator

over the points of constant energy $\varepsilon_\nu(\kappa_x) = \varepsilon$, thus

$$\hat{I}F_{0\nu}(\varepsilon) = \frac{\sum_{\kappa_x} \hat{I}F_{0\nu}(\varepsilon_\nu(\kappa_x))\delta(\varepsilon_\nu(\kappa_x) - \varepsilon)}{\sum_{\kappa_x} \delta(\varepsilon_\nu(\kappa_x) - \varepsilon)}. \tag{4.1}$$

This procedure gives the expression

$$\begin{aligned}\hat{I}F_{0\nu}(\varepsilon) = \frac{1}{N_\nu(\varepsilon)}\frac{1}{L_x}\sum_{\nu'}\sum_{\kappa_x}\sum_{\boldsymbol{q}} w(q)G^2_{\nu\nu'}(\boldsymbol{q}_\perp)\{\{F_{0\nu'}[\varepsilon_{\nu'}(\kappa_x+q_x)](N_{\boldsymbol{q}}+1) - F_{0\nu}(\varepsilon)N_{\boldsymbol{q}}\} \\ \times\delta[\varepsilon_{\nu'}(\kappa_x+q_x) - \varepsilon - \overline{h}sq] + \{F_{0\nu'}[\varepsilon_{\nu'}(\kappa_x+q_x)]N_{\boldsymbol{q}} - F_{0\nu}(\varepsilon)(N_{\boldsymbol{q}}+1)\} \\ \times\delta[\varepsilon_{\nu'}(\kappa_x+q_x) - \varepsilon + \overline{h}sq]\}\delta(\varepsilon_\nu(\kappa_x) - \varepsilon)\end{aligned} \tag{4.2}$$

where we have used the notation $N_\nu(\varepsilon)$ for the electron density of states. If the electron kinetic energy in sub-band ν is within the range defined by equation (3.1) ($\varepsilon_\parallel > \overline{h}sq_{\perp 0}$) then the electron–phonon interaction is quasi-elastic for intra- and inter-sub-band transitions. The phonon energy $\overline{h}sq$ is small compared with the electron kinetic energy $\varepsilon_\parallel(\kappa_x) = \varepsilon - W_\nu$ and we can expand the delta-functions in equation (4.2) in a Taylor series:

$$\delta[\varepsilon_{\nu'}(\kappa_x+q_x) - \varepsilon \mp \overline{h}sq] = \left(1 \pm \overline{h}sq\frac{\mathrm{d}}{\mathrm{d}\varepsilon} + \frac{1}{2}(\overline{h}sq)^2\frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2}\right)\delta[\varepsilon_{\nu'}(\kappa_x+q_x) - \varepsilon]. \tag{4.3}$$

With these substitutions $\hat{I}F_{0\nu}(\varepsilon)$ becomes

$$\begin{aligned}\hat{I}F_{0\nu}(\varepsilon) = \frac{1}{N_\nu(\varepsilon)}\frac{1}{L_x}\sum_{\nu'}\sum_{\kappa_x}\delta(\varepsilon_\nu(\kappa_x) - \varepsilon)\sum_{\boldsymbol{q}} w(q)G^2_{\nu\nu'}(\boldsymbol{q}_\perp) \\ \times\{[F_{0\nu'}(\varepsilon) - F_{0\nu}(\varepsilon)](2N_{\boldsymbol{q}}+1)\delta[\varepsilon_{\nu'}(\kappa_x+q_x) - \varepsilon] \\ +\{F_{0\nu'}[\varepsilon_{\nu'}(\kappa_x+q_x)] + F_{0\nu}(\varepsilon)\}\overline{h}sq\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\delta[\varepsilon_{\nu'}(\kappa_x+q_x) - \varepsilon] \\ +\{F_{0\nu'}[\varepsilon_{\nu'}(\kappa_x+q_x)] - F_{0\nu}(\varepsilon)\}2N_{\boldsymbol{q}}+1)\frac{1}{2}(\overline{h}sq)^2\frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2}\delta[\varepsilon_{\nu'}(\kappa_x+q_x) - \varepsilon]\}.\end{aligned} \tag{4.4}$$

Further analytical transformation of this expression can be performed only by making some additional simplifying assumptions concerning the phonon distribution function $N_{\boldsymbol{q}}$.

4.1. The equipartition approximation

The first natural approximation is based on the equipartition approximation

$$\overline{h}sq < T_0 \tag{4.5}$$

$$N_{\boldsymbol{q}} \simeq \frac{T_0}{\overline{h}sq}. \tag{4.6}$$

The case of a rectangular QWI was investigated in [5]. Here we give a more general treatment. The condition from equation (4.5) holds for the phonons with which the carriers interact if the lattice temperature T_0 is defined by $T_0 > \overline{h}sq_{\perp 0}$. In fact, in accordance with paper I (equation (5.24) in I) for the electron kinetic energy $\varepsilon_\parallel(\kappa_x) < W_0$ we have $q_\perp > |q_x|$ and $q \simeq q_\perp \leqslant q_{\perp 0}$, hence equation (4.5) is realized automatically if $T_0 > \overline{h}sq_{\perp 0}$ holds. If the electron kinetic energy is $\varepsilon_\parallel(\kappa_x) > W_0$ then $|q_x| > q_\perp$ and $q \simeq |q_x| \simeq 2|\kappa_x|$. Substitution of this value q into equation (4.5) gives us the upper limit for the electron kinetic energy $\varepsilon_\parallel(\kappa_x) < T_0^2/(8m^*s^2)$ to maintain the validity of equation (4.5). Therefore,

we can state that the equipartition approximation of equation (4.6) is realized within the high-lattice-temperature region defined by $T_0 > \overline{h}sq_{\perp 0}$ and for the electron kinetic energy given by

$$\varepsilon_{\parallel}(\kappa_x) < \frac{T_0^2}{8m^*s^2}. \tag{4.7}$$

This is true as well for the inter-sub-band scattering, as follows from paper I (equations (5.27) and (5.28) in I).

On combining equations (3.1) and (4.7), we obtain the range of the electron energy within which both the equipartition and the quasi-elastic approximations are realized:

$$(8m^*s^2W_0)^{1/2} < \varepsilon - w_\nu < \frac{T_0^2}{8m^*s^2}. \tag{4.8}$$

Note that, in accordance with equations (2.5) and (4.6), the equipartition approximation is the case in which the product $w(q)(2N_q + 1)$ does not depend on q_x, so we can use this property for transformation of the antisymmetric collision operator $\hat{I}F_\nu^-(\kappa_x)$.

Making use of the approximation from equation (4.6), we can present equation (4.4) in the form (see the appendix)

$$\hat{I}F_{0\nu}(\varepsilon) = \frac{4m^*s}{\overline{h}}w_0L_x\sum_{\nu'}\frac{1}{N_{\nu'}(\varepsilon)}\left\{\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[N_{\nu'}^2(\varepsilon)A_{\nu\nu'}(\varepsilon)\left(F_{0\nu'}(\varepsilon) + T_0\frac{\mathrm{d}F_{0\nu'}(\varepsilon)}{\mathrm{d}\varepsilon}\right)\right]\right.$$
$$\left. + \frac{T_0}{2m^*s^2}N_{\nu'}^2(\varepsilon)B_{\nu\nu'}(\varepsilon)[F_{0\nu'}(\varepsilon) - F_{0\nu}(\varepsilon)]\right\}. \tag{4.9}$$

Here we have introduced the following notations:

$$A_{\nu\nu'}(\varepsilon) = \frac{1}{2}\frac{\displaystyle\sum_{\kappa_x}\sum_{\boldsymbol{q}}G_{\nu\nu'}^2(\boldsymbol{q}_\perp)\frac{\overline{h}^2q^2}{2m^*}\delta(\varepsilon_\nu(\kappa_x) - \varepsilon)\delta[\varepsilon_{\nu'}(\kappa_x + q_x) - \varepsilon]}{\displaystyle\sum_{\kappa_x}\sum_{q_x}\delta(\varepsilon_\nu(\kappa_x) - \varepsilon)\delta[\varepsilon_{\nu'}(\kappa_x + q_x) - \varepsilon]} \tag{4.10}$$

$$B_{\nu\nu'}(\varepsilon) = \frac{\displaystyle\sum_{\kappa_x}\sum_{\boldsymbol{q}}G_{\nu\nu'}^2(\boldsymbol{q}_\perp)\delta(\varepsilon_\nu(\kappa_x) - \varepsilon)}{\displaystyle\sum_{\kappa_x}\sum_{q_x}\delta(\varepsilon_\nu(\kappa_x) - \varepsilon)\delta[\varepsilon_{\nu'}(\kappa_x + q_x) - \varepsilon]}$$
$$\times\left(1 - \frac{(\overline{h}sq)^2}{2T_0}\frac{\mathrm{d}}{\mathrm{d}\varepsilon} + \frac{(\overline{h}sq)^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2}\right)\delta[\varepsilon_{\nu'}(\kappa_x + q_x) - \varepsilon]. \tag{4.11}$$

As one can see, equations (4.9)–(4.11) hold for an arbitrary shape of the quantizing potential. All dependence of the symmetric collision operator $\hat{I}F_\nu(\varepsilon)$ on this potential is described in terms of the form factor $G_{\nu\nu'}^2(\varepsilon)$ and the electron energy spectrum $\varepsilon_\nu(\kappa_x)$ which are included in the coefficients $A_{\nu\nu'}(\varepsilon)$ and $B_{\nu\nu'}(\varepsilon)$.

For rectangular 1D QWI the form factor $G_{\nu\nu'}^2(\boldsymbol{q}_\perp)$ is given by equations (4.11) and (4.12) in paper I. To calculate the coefficients $A_{\nu\nu'}(\varepsilon)$ and $B_{\nu\nu'}(\varepsilon)$ we can use the following values of the sums:

$$\sum_{\boldsymbol{q}_\perp}G_{\nu\nu'}^2(\boldsymbol{q}_\perp) = \left(1 + \frac{\delta_{nn'}}{2}\right)\left(1 + \frac{\delta_{pp'}}{2}\right) \tag{4.12}$$

$$\sum_{\boldsymbol{q}_\perp}G_{\nu\nu'}^2(\boldsymbol{q}_\perp)\boldsymbol{q}_\perp^2 = \frac{\pi^2}{L_y^2}\left(1 + \frac{\delta_{pp'}}{2}\right)(n^2 + n'^2) + \frac{\pi^2}{L_z^2}\left(1 + \frac{\delta_{nn'}}{2}\right)(p^2 + p'^2). \tag{4.13}$$

The coefficient $B_{\nu\nu'}(\varepsilon)$ in equation (4.9) describes the contribution of the inter-sub-band scattering only to the relaxation of the distribution function $F_{0\nu}(\varepsilon)$. The second term in equation (4.9) contains difference $F_{0\nu'}(\varepsilon) - F_{0\nu}(\varepsilon)$ and it vanishes when $\nu' = \nu$. We can neglect in the square brackets of equation (4.11) the second and third terms compared with the first because the electron energy is within the range defined by equation (4.8) and the lattice temperature is given by $T_0 > \hbar s q_{\perp 0}$. As a result these terms are small. This means that we neglect the quasi-elastic contribution to the inter-sub-band relaxation compared with what is effectively the purely elastic contribution. It is obvious physically that the symmetrical part of the distribution function $F_{0\nu}(\varepsilon)$ in sub-band ν will be influenced by the inter-sub-band transitions even in the elastic approximation, which is of course impossible for the intra-sub-band elastic scattering.

By using equations (4.12) and (4.13) we obtain for the coefficients $A_{\nu\nu'}(\varepsilon)$ and $B_{\nu\nu'}(\varepsilon)$

$$A_{\nu\nu'}(\varepsilon) = \varepsilon\left(1+\frac{\delta_{nn'}}{2}\right)\left(1+\frac{\delta_{pp'}}{2}\right) - \frac{\pi^2\hbar^2}{2m^*}\left[\frac{n^2}{L_y^2}\frac{\delta_{nn'}}{2}\left(1+\frac{\delta_{pp'}}{2}\right) + \frac{p^2}{L_z^2}\frac{\delta_{pp'}}{2}\left(1+\frac{\delta_{pp'}}{2}\right)\right] \tag{4.14}$$

$$B_{\nu\nu'}(\varepsilon) = \left(1+\frac{\delta_{nn'}}{2}\right)\left(1+\frac{\delta_{pp'}}{2}\right). \tag{4.15}$$

This gives the following final expression for the symmetric collision operator:

$$\begin{aligned}\hat{I}F_{0\nu}(\varepsilon) = \frac{4m^*s}{\hbar}w_0T_0L_x\sum_{\nu'}\frac{1}{N_{\nu'}(\varepsilon)}\Bigg\{\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Bigg\{N_{\nu'}^2(\varepsilon)\Bigg\{\varepsilon\left(1+\frac{\delta_{nn'}}{2}\right)\left(1+\frac{\delta_{pp'}}{2}\right)\\ -\frac{\pi^2\hbar^2}{2m^*}\left[\frac{n^2}{L_y^2}\left(1+\frac{\delta_{pp'}}{2}\right)+\frac{p^2}{L_z^2}\left(1+\frac{\delta_{nn'}}{2}\right)\right]\Bigg\}\left(\frac{\mathrm{d}F_{0\nu'}(\varepsilon)}{\mathrm{d}\varepsilon}+\frac{1}{T_0}F_{0\nu'}(\varepsilon)\right)\Bigg\}\\ +\frac{1}{2m^*s^2}N_{\nu'}^2(\varepsilon)\left(1+\frac{\delta_{nn'}}{2}\right)\left(1+\frac{\delta_{pp'}}{2}\right)[F_{0\nu'}(\varepsilon)-F_{0\nu}(\varepsilon)]\Bigg\}.\end{aligned} \tag{4.16}$$

The ranges of electron energy and the lattice temperature for which equation (4.16) holds are just those given by $(8m^*s^2W_0)^{1/2} < \varepsilon - W_\nu < T_0^2/(8m^*s^2)$ and $T_0 > (8m^*s^2W_0)^{1/2}$, respectively.

4.2. The zero-point lattice approximation

Let us now investigate the low-lattice-temperature case for the same electron energy range as above defined by equation (3.1). We will assume that

$$\hbar s q > T_0 \tag{4.17}$$

for the majority of phonons with which the carriers interact. This also means that the lattice temperature is within the range $T_0 < \hbar s q_{\perp 0}$. Because of the deformation acoustic potential in equation (4.4) $w(q) \propto q$, the main contribution to the sum over $\boldsymbol{q}$ comes from the region of large q. The phonon distribution function $N_{\boldsymbol{q}}$ appears in equation (4.4) through the expression $(2N_{\boldsymbol{q}} + 1)$ and, as was shown above, if the lattice temperature is defined by $T_0 < \hbar s q_{\perp 0}$ then, for the terms which correspond to the stimulated scattering processes and which contain $N_{\boldsymbol{q}}$, the region of actual $\boldsymbol{q}$ is restricted to $q \lesssim T_0/(\hbar s)$. The other term, which corresponds to the spontaneous emission, does not contain $N_{\boldsymbol{q}}$ and thus the region of actual $\boldsymbol{q}$ is restricted to $q_\perp \lesssim q_{\perp 0}$. The first restriction, $q \lesssim T_0/(\hbar s)$, is more severe than the second one, $q_\perp \lesssim q_{\perp 0}$, if the lattice temperature is defined by $T_0 < \hbar s q_{\perp 0}$. This means that, under these conditions, the region of $\boldsymbol{q}$-space which contributes to the stimulated scattering processes is small compared with that which contributes to the spontaneous

scattering processes. As a result we can use the zero-point lattice approximation and put in equation (4.4)

$$N_{\boldsymbol{q}} \simeq 0. \tag{4.18}$$

Equation (4.4) can be transformed into the following expression (see the appendix for details):

$$\begin{aligned}\hat{I}F_{0\nu}(\varepsilon) = \frac{4m^*s}{\overline{h}} w_0 L_x \sum_{\nu'} \frac{1}{N_{\nu'}(\varepsilon)} \left\{ \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \left[N^2_{\nu'}(\varepsilon) \left(A_{\nu\nu'}(\varepsilon) F_{0\nu'}(\varepsilon) + C_{\nu\nu'}(\varepsilon) \frac{\mathrm{d}F_{0\nu'}(\varepsilon)}{\mathrm{d}\varepsilon} \right) \right] \right. \\ + N^2_{0\nu'}(\varepsilon) \left(\frac{1}{2m^*s^2} D_{\nu\nu'}(\varepsilon)[F_{\nu'}(\varepsilon) - F_{\nu}(\varepsilon)] \right. \\ \left. \left. - \Delta_{\nu\nu'}(\varepsilon) \frac{W_\nu - W_{\nu'}}{[(\varepsilon - W_\nu)(\varepsilon - W_{\nu'})]^{1/2}} \frac{\mathrm{d}F_{0\nu'}(\varepsilon)}{\mathrm{d}\varepsilon} \right) \right\}\end{aligned} \tag{4.19}$$

where

$$C_{\nu\nu'}(\varepsilon) = \frac{1}{4} \frac{\displaystyle\sum_{\kappa_x} \sum_{\boldsymbol{q}} (\overline{h}^2 q^2/2m) \overline{h} sq G^2_{\nu\nu'}(\boldsymbol{q}_\perp) \delta(\varepsilon_\nu(\kappa_x) - \varepsilon) \delta[\varepsilon_{\nu'}(\kappa_x + q_x) - \varepsilon]}{\displaystyle\sum_{\kappa_x} \sum_{q_x} \delta(\varepsilon_\nu(\kappa_x) - \varepsilon) \delta[\varepsilon_{\nu'}(\kappa_x + q_x) - \varepsilon]} \tag{4.20}$$

$$\begin{aligned}D_{\nu\nu'}(\varepsilon) = \frac{1}{2} \sum_{\kappa_x} \sum_{\boldsymbol{q}} \overline{h} sq G^2_{\nu\nu'}(\boldsymbol{q}_\perp) \delta(\varepsilon_\nu(\kappa_x) - \varepsilon) \left\{ \left(1 - \overline{h} sq \frac{\mathrm{d}}{\mathrm{d}\varepsilon} + \frac{1}{2} (\overline{h} sq)^2 \frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2} \right) \right. \\ \left. \times \delta[\varepsilon_{\nu'}(\kappa_x + q_x) - \varepsilon] \right\} \left\{ \sum_{\kappa_x} \sum_{q_x} \delta(\varepsilon_\nu(\kappa_x) - \varepsilon) \delta[\varepsilon_{\nu'}(\kappa_x + q_x) - \varepsilon] \right\}^{-1}\end{aligned} \tag{4.21}$$

$$\Delta_{\nu\nu'}(\varepsilon) = \frac{3}{16} (2m^*s^2)^{1/2} \left[\bar{G}^+_{\nu\nu'} \left(\varepsilon, \frac{1}{2} \right) - \bar{G}^-_{\nu\nu'} \left(\varepsilon, \frac{1}{2} \right) \right] \tag{4.22}$$

$$\bar{G}^\pm_{\nu\nu'} \left(\varepsilon, \frac{n}{2} \right) = \sum_{\boldsymbol{q}\perp} G^2_{\nu\nu'}(\boldsymbol{q}_\perp) \left(\frac{\overline{h}^2 q_\perp^2}{2m^*} + [(\varepsilon - W_\nu)^{1/2} \pm (\varepsilon - W_{\nu'})^{1/2}]^2 \right)^{n/2}. \tag{4.23}$$

Equations (4.19)–(4.23) are true for an arbitrary shape of the quantizing potential, as were similar equations (4.9)–(4.11) for the equipartition approximation. However, there is a difference between equations (4.9) and (4.11). As one can see, the second term in equation (4.4) does not depend on $N_{\boldsymbol{q}}$ and is the same for both approximations (equipartition and zero point). The first term in equation (4.4), which is proportional to $F_{\nu'}(\varepsilon) - F_\nu(\varepsilon)$, is transformed in the same manner for both cases. There is only an additional factor $\overline{h} sq$ in the coefficient $D_{\nu\nu'}(\varepsilon)$ compared with the coefficient $B_{\nu\nu'}(\varepsilon)$. Note that here we can also neglect the second and the third terms in the square brackets of equation (4.21). Transformation of the third term in equation (4.4) gives different general forms for the discussed approximations. It is shown in the appendix for the equipartition approximation that this term has the same structure as the second one (see equation (4.9)), but not for the zero-point lattice approximation. The function $\Phi_{\nu\nu'}(\boldsymbol{q}_\perp, q_x)$ of equation (A3) in the last case is equal to $\Phi_{\nu\nu'}(\boldsymbol{q}_\perp, q_x) = \frac{1}{2} w(q) G^2_{\nu\nu'}(\boldsymbol{q}_\perp)(\overline{h} sq)^2$ and the relationship in equation (A2) does not hold for this function when $\nu \neq \nu'$. As a result we have an additional term in equation (4.19) which is absent from equation (4.9).

The coefficients $C_{\nu\nu'}(\varepsilon)$ and $D_{\nu\nu'}(\varepsilon)$ can be presented in the form

$$C_{\nu\nu'}(\varepsilon) = \frac{1}{8} (2m^*s^2)^{1/2} \left[\bar{G}^+_{\nu\nu'} \left(\varepsilon, \frac{3}{2} \right) + \bar{G}^-_{\nu\nu'} \left(\varepsilon, \frac{3}{2} \right) \right] \tag{4.24}$$

$$D_{\nu\nu'}(\varepsilon) = \frac{1}{4}(2m^*s^2)^{1/2}\left[\bar{G}^{+}_{\nu\nu'}\left(\varepsilon, \frac{1}{2}\right) + \bar{G}^{-}_{\nu\nu'}\left(\varepsilon, \frac{1}{2}\right)\right]. \tag{4.25}$$

For the rectangular 1D QWI the form factor $G^2_{\nu\nu'}(\boldsymbol{q}_\perp)$ is given in paper I (equations (4.11) and (4.12) in I) and the summation over $\boldsymbol{q}_\perp$ in equations (4.22)–(4.24) cannot be performed analytically, unlike for the equipartition case, in which we have used equations (4.12) and (4.13). Due to the presence of fractional powers like $(q_\perp^2+\alpha_0)^{n/2}$ (where $n = 1, 3$ and α_0 is some constant) in these expressions, the analytical calculations can be performed only for some limiting cases. We will consider these cases when we solve the Boltzmann equation.

Regarding the generality of the above expressions for the symmetrical collision operator $\hat{I}F_{0\nu}(\varepsilon)$, we have to point out that equations (4.9) and (4.19) will remain true (after obvious modification) for a 2D electron gas interacting with acoustic phonons. (The factor $L_x/(2\pi)$ must be replaced everywhere by $L_xL_y/(2\pi)^2$ (quantization along the z axis is assumed) and wavevectors κ_x and q_x must be replaced by $\boldsymbol{\kappa}_\parallel \equiv (\kappa_x, \kappa_y)$ and $\boldsymbol{q}_\parallel \equiv (q_x, q_y)$, respectively).

If we take into account only intra-sub-band processes ($\nu = \nu'$), which is valid if electrons populate only the first sub-band, then $\hat{I}F_{0\nu}(\varepsilon)$ can be presented in the differential form without making any assumption concerning the phonon distribution function $N_{\boldsymbol{q}}$. By making use of the relationship equation (A2), we obtain from equation (4.4)

$$\hat{I}F_{0\nu}(\varepsilon) = \frac{4m^*s}{\hbar}w_0L_x\frac{1}{N_\nu(\varepsilon)}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[N_\nu^2(\varepsilon)\left(A_{\nu\nu}(\varepsilon)F_{0\nu}(\varepsilon) + C'_{\nu\nu}(\varepsilon)\frac{\mathrm{d}F_{0\nu}(\varepsilon)}{\mathrm{d}\varepsilon}\right)\right] \tag{4.26}$$

where $A_{\nu\nu}(\varepsilon)$ is given by equation (4.10) and

$$C'_{\nu\nu}(\varepsilon) = \frac{1}{4}\frac{\displaystyle\sum_{\kappa_x}\sum_{\boldsymbol{q}} G^2_{\nu\nu}(\boldsymbol{q}_\perp)(2N_{\boldsymbol{q}}+1)(\hbar^2q^2/2m^*)\hbar sq\,\delta(\varepsilon_\nu(\kappa_x)-\varepsilon)\delta[\varepsilon_\nu(\kappa_x+q_x)-\varepsilon]}{\displaystyle\sum_{\kappa_x}\sum_{q_x}\delta(\varepsilon_\nu(\kappa_x)-\varepsilon)\delta[\varepsilon_\nu(\kappa_x+q_x)-\varepsilon]}. \tag{4.27}$$

One can see that the coefficient $C_{\nu\nu'}(\varepsilon)$ coincides with the corresponding expressions in equations (4.9) or (4.19) for the discussed limiting cases. The expression in equation (4.26) is valid for 1D, 2D and 3D electron gases. For the 3D case it is necessary to replace κ_x and q_x by $\boldsymbol{\kappa}$ and $\boldsymbol{q}$, respectively, and $L_x/(2\pi)$ by $V_0/(2\pi)^3$ in equations (4.10), (4.26) and (4.27), and to put formally $G^2_{\nu\nu}(\boldsymbol{q}) = 1$ and $W_0 = 0$.

5. The form of the symmetric collision operator for inelastic scattering

If the electron kinetic energy in sub-band ν is within the range defined by equation (3.8) ($\varepsilon_\parallel < \hbar sq_{\perp 0}$) then the relaxation of the symmetrical distribution function $F_{0\nu}(\varepsilon)$ is described both by the inelastic intra-sub-band scattering and by the quasi-elastic inter-sub-band scattering. It is obvious that, in this case, only sub-bands with $\nu' < \nu$ are involved in the relaxation of $F_{0\nu}(\varepsilon)$ and for these sub-bands the electron kinetic energy is within the range

$$\varepsilon_{\nu'}(\kappa_x) - W_{\nu'} > (8m^*s^2W_0)^{1/2} \qquad (\nu' < \nu). \tag{5.1}$$

This is why the electron–acoustic-phonon interaction is quasi-elastic for inter-sub-band scattering.

We can therefore present the symmetric collision operator $\hat{I}F_{0\nu}(\varepsilon)$ in equation (4.2) in the form

$$\hat{I}F_{0\nu}(\varepsilon) = \hat{I}_{inel}F_{0\nu}(\varepsilon) + \hat{I}_{quasi\text{-}el}F_{0\nu}(\varepsilon) \tag{5.2}$$

where

$$\hat{I}_{inel}F_{0\nu}(\varepsilon) + \hat{I}_{\nu\nu}F_{0\nu}(\varepsilon) \tag{5.3}$$

$$\hat{I}_{quasi\text{-}el}F_{0\nu}(\varepsilon) = \sum_{\nu'<\nu}\hat{I}_{\nu\nu'}F_{0\nu}(\varepsilon) \tag{5.4}$$

$$\begin{aligned}\hat{I}_{\nu\nu'}F_{0\nu}(\varepsilon) = \frac{1}{N_\nu(\varepsilon)}\frac{1}{L_x}\sum_{\kappa_x}\sum_{\boldsymbol{q}} w(q)G^2_{\nu\nu'}(q_\perp)\{\{F_{0\nu'}[\varepsilon_{\nu'}(\kappa_x+q_x)](N_{\boldsymbol{q}}+1) - F_{0\nu}(\varepsilon)N_{\boldsymbol{q}}\}\\ \times\delta[\varepsilon_{\nu'}(\kappa_x+q_x) - \varepsilon - \overline{\hbar}sq] + \{F_{0\nu'}[\varepsilon_{\nu'}(\kappa_x+q_x)]N_{\boldsymbol{q}} - F_{0\nu}(\varepsilon)(N_{\boldsymbol{q}}+1)\}\\ \times\delta[\varepsilon_{\nu'}(\kappa_x+q_x) - \varepsilon + \overline{\hbar}sq)]\}\delta(\varepsilon_\nu(\kappa_x) - \varepsilon).\end{aligned} \tag{5.5}$$

The expressions for the quasi-elastic collision operator were derived in section 4.

To calculate the inelastic collision operator $\hat{I}_{inel}F_{0\nu}(\varepsilon)$ we can use the condition $q_\perp \gg |q_x|$ (see equation (5.5) in paper I) and put $q \cong q_\perp$ in equations (5.3) and (5.5) for $\nu' = \nu$. On performing the summation over q_x and κ_x we obtain

$$\begin{aligned}\hat{I}_{inel}F_{0\nu}(\varepsilon) = \frac{V_0}{(2\pi)^2}w_0\Bigg(\int_0^\infty N_\nu(\varepsilon+\overline{\hbar}sq_\perp)G^2_{\nu\nu}(\boldsymbol{q}_\perp)q_\perp\\ \times[F_{0\nu}(\varepsilon+\overline{\hbar}sq_\perp)(N_{\boldsymbol{q}_\perp}+1) - F_{0\nu}(\varepsilon)N_{\boldsymbol{q}_\perp}]\,\mathrm{d}\boldsymbol{q}_\perp\\ + \int_0^{(\varepsilon-W_\nu)/(\overline{\hbar}s)} N_\nu(\varepsilon-\overline{\hbar}sq_\perp)G^2_{\nu\nu}(\boldsymbol{q}_\perp)q_\perp\\ \times[F_{0\nu}(\varepsilon-\overline{\hbar}sq_\perp)N_{\boldsymbol{q}_\perp} - F_{0\nu}(\varepsilon)(N_{\boldsymbol{q}_\perp}+1)]\,\mathrm{d}\boldsymbol{q}_\perp\Bigg).\end{aligned} \tag{5.6}$$

The upper limit in the second integral in equation (5.6) is defined by the condition that the density of states $N_\nu(\varepsilon - \overline{\hbar}sq_\perp)$ has to be real and positive. Because this term includes the scattering from the κ_x state of the electron in sub-band ν due to the emission of the acoustic phonon, it is obvious physically that the electron cannot emit a phonon with energy that is larger than its own kinetic energy, $\overline{\hbar}sq_\perp \leqslant \varepsilon_\parallel(k_x) = \varepsilon - W_\nu$, in agreement with the upper limit in equation (5.6).

Note that, if the electron energy is defined by equation (3.8) and we can use the approximation $q \cong q_\perp$, then this is the case when the products $w(q)N_{\boldsymbol{q}}$ and $w(q)(N_{\boldsymbol{q}}+1)$ are not depend on q_x. We have applied this fact in section 2 to introduce the momentum relaxation time approximation for the antisymmetric collision operator $\hat{I}F^-_\nu(\kappa_x)$.

It follows from equation (5.6) that the symmetric collision operator $\hat{I}_{inel}F_{0\nu}(\varepsilon)$ has an integral form in contradistinction to the differential form in equations (4.9) and (4.19). This is the direct result of the inelasticity of the electron–acoustic-phonon interaction under the discussed conditions. Since the characteristic phonon energy, $\overline{\hbar}\omega_{\boldsymbol{q}} \lesssim \overline{\hbar}sq_{\perp 0}$, has the same value as the electron kinetic energy $\varepsilon_\parallel(\kappa_x) \lesssim (8m^*s^2W_0)^{1/2}$, we cannot use the expansions performed in equation (4.3).

When the electron kinetic energy is large enough to be within the range

$$(8m^*s^2W_0)^{1/2} < \varepsilon_\nu(\kappa_x) - W_\nu < W_0 \tag{5.7}$$

we can use the approximation $q \simeq q_\perp$ as before. This means that equation (5.6) remains valid as well. However, due to presence of the form factor in the integrand we can put the upper limit in the second integral to infinity. Furthermore, we can use the quasi-elastic approximation and expand the functions $F_{0\nu}(\varepsilon \pm \overline{\hbar}sq_\perp)$ in a Taylor series. As a result equation (5.6) is transformed into the differential expression in equation (4.26), where it is necessary to put $q \simeq q_\perp$.

If the electron kinetic energy is within the range

$$\varepsilon_\nu(\kappa_x) - W_\nu > W_0 \tag{5.8}$$

then we cannot use the approximation $q \cong q_\perp$. Equation (5.6) becomes invalid and instead we have to use equation (4.26). All this means that it is necessary to divide the symmetrical collision operator $\hat{I}F_{0\nu}(\varepsilon)$ into the inelastic and quasi-elastic contributions as in equation (5.2) only if the electron kinetic energy is within the interval defined by equation (3.8). It is relevant to note here that equation (5.6) is true for a 2D electron gas under the same conditions as discussed above.

Now, after detailed investigation of the symmetrical and antisymmetrical collision operators, we can solve the Boltzmann equations (2.11) and (2.12) for some cases of interest. Since the forms of these operators are different under different conditions we will consider in the following paper III separately the situations of inelastic and quasi-elastic scattering.

6. Summary

In this paper we derived the Boltzmann equation for a non-equilibrium electron gas in a low-dimensional system interacting with bulk acoustic phonons. In the general case this equation has a form which depends on the external conditions and the electron energy region in which the electron distribution is studied.

At low lattice temperatures the Boltzmann equation has an integro-differential form within a wide range of external electric fields. This is a direct result of the strong inelastic character of the electron–acoustic-phonon interaction under these conditions. The actual energy region lies within the first electron sub-band. For larger electric fields, for which the electrons penetrate into high-energy regions, their interaction with acoustic phonons becomes quasi-elastic. As a result the Boltzmann equation has a differential form. This is true both for intra- and for inter-sub-band scattering.

At high lattice temperatures the electron–acoustic-phonon interaction is always quasi-elastic for the majority of electrons at arbitrary electric fields. The corresponding Boltzmann equation consists of a set of differential equations for the distribution functions in every occupied electron sub-band.

An important question for the electron kinetics in a low-dimensional system is when to use the momentum-relaxation-time approximation for the antisymmetrical collision operator. We performed a detailed investigation of this question and we have shown that it is possible to introduce the momentum relaxation time both for strong inelastic and for quasi-elastic scattering. The corresponding expressions for the momentum relaxation times were obtained.

An important result was the derivation of a general form for the Boltzmann equation for low-dimensional electron systems. The final expressions can be applied to three-, two- or one-dimensional electron gases subjected to an arbitrary quantizing potential. As an example we applied these expressions to the case of a one-dimensional rectangular quantum well wire.

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Appendix A. Transformation of equation (4.4) into equation (4.9)

Let us consider separately the second and the third terms in equation (4.4).

We have for the second term

$$\begin{aligned}
&\sum_{\kappa_x}\delta(\varepsilon_\nu(\kappa_x)-\varepsilon)\sum_{\boldsymbol{q}}w(q)G^2_{\nu\nu'}(\boldsymbol{q}_\perp)\overline{h}sq\{F_{0\nu'}[\varepsilon_{\nu'}(\kappa_x+q_x)]+F_{0\nu}(\varepsilon)\}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\delta[\varepsilon_{\nu'}(\kappa_x+q_x)-\varepsilon]\\
&\quad=\sum_{\kappa_x}\delta(\varepsilon_\nu(\kappa_x)-\varepsilon)\\
&\qquad\times\left[\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left(\sum_{\boldsymbol{q}}w(q)G^2_{\nu\nu'}(\boldsymbol{q}_\perp)\overline{h}sqF_{0\nu'}[\varepsilon_{\nu'}(\kappa_x+q_x)]\delta[\varepsilon_{\nu'}(\kappa_x+q_x)-\varepsilon]\right)\right.\\
&\qquad\left.+F_{0\nu'}(\varepsilon)\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left(\sum_{\boldsymbol{q}}w(q)G^2_{\nu\nu'}(\boldsymbol{q}_\perp)\overline{h}sq[\varepsilon_{\nu'}(\kappa_x+q_x)-\varepsilon]\right)\right]\\
&\quad=\sum_{\kappa_x}\delta(\varepsilon_\nu(\kappa_x)-\varepsilon)\\
&\qquad\times\left[[F_{0\nu'}(\varepsilon)-F_{0\nu}(\varepsilon)]\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left(\sum_{\boldsymbol{q}}w(q)G^2_{\nu\nu'}(\boldsymbol{q}_\perp)\overline{h}sq\delta[\varepsilon_{\nu'}(\kappa_x+q_x)-\varepsilon]\right)\right.\\
&\qquad+\frac{\mathrm{d}F_{0\nu'}(\varepsilon)}{\mathrm{d}\varepsilon}\sum_{\boldsymbol{q}}w(q)G^2_{\nu\nu'}(\boldsymbol{q}_\perp)\overline{h}sq\delta[\varepsilon_{\nu'}(\kappa_x+q_x)-\varepsilon]\\
&\qquad\left.+2F_{0\nu'}(\varepsilon)\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left(\sum_{\boldsymbol{q}}w(q)G^2_{\nu\nu'}(\boldsymbol{q}_\perp)\overline{h}sq\delta[\varepsilon_{\nu'}(\kappa_x+q_x)-\varepsilon]\right)\right]\\
&\quad=-[F_{0\nu'}(\varepsilon)-F_{0\nu}(\varepsilon)]\sum_{\kappa_x}\sum_{\boldsymbol{q}}w(q)G^2_{\nu\nu'}(\boldsymbol{q}_\perp)\overline{h}sq\delta(\varepsilon_\nu(\kappa_x)-\varepsilon)\\
&\qquad\times\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\delta[\varepsilon_{\nu'}(\kappa_x+q_x)-\varepsilon]\\
&\qquad+\left[\frac{\mathrm{d}F_{0\nu'}(\varepsilon)}{\mathrm{d}\varepsilon}\left(\sum_{\kappa_x}\sum_{\boldsymbol{q}}w(q)G^2_{\nu\nu'}(\boldsymbol{q}_\perp)\overline{h}sq\delta(\varepsilon_\nu(\kappa_x)-\varepsilon)\delta[\varepsilon_{\nu'}(\kappa_x+q_x)-\varepsilon]\right)\right.\\
&\qquad\left.+F_{0\nu'}(\varepsilon)\left(2\sum_{\kappa_x}\sum_{\boldsymbol{q}}w(q)G^2_{\nu\nu'}(\boldsymbol{q}_\perp)\overline{h}sq\delta(\varepsilon_\nu(\kappa_x)-\varepsilon)\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\delta[\varepsilon_{\nu'}(\kappa_x+q_x)-\varepsilon]\right)\right]\\
&\quad=-[F_{0\nu'}(\varepsilon)-F_{0\nu}(\varepsilon)]\sum_{\kappa_x}\sum_{\boldsymbol{q}}w(q)G^2_{\nu\nu'}(\boldsymbol{q}_\perp)\overline{h}sq\delta(\varepsilon_\nu(\kappa_x)-\varepsilon)\\
&\qquad\times\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\delta[\varepsilon_{\nu'}(\kappa_x+q_x)-\varepsilon]\\
&\qquad+\frac{N_\nu(\varepsilon)}{N_{\nu'}(\varepsilon)}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left(\frac{N_{\nu'}(\varepsilon)}{N_\nu(\varepsilon)}\sum_{\kappa_x}\sum_{\boldsymbol{q}}w(q)G^2_{\nu\nu'}(\boldsymbol{q}_\perp)\overline{h}sq\delta(\varepsilon_\nu(\kappa_x)-\varepsilon)\right.\\
&\qquad\left.\times\delta[\varepsilon_{\nu'}(\kappa_x+q_x)-\varepsilon]F_{0\nu'}(\varepsilon)\right). \qquad \text{(A1)}
\end{aligned}$$

Here in the final stage of the transformation we have used the following relationship:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left(\frac{N_{\nu'}(\varepsilon)}{N_\nu(\varepsilon)}\sum_{\kappa_x}\sum_{\boldsymbol{q}}\Phi_{\nu\nu'}(\boldsymbol{q}_{\perp'}q_x)\delta(\varepsilon_\nu(\kappa_x)-\varepsilon)\delta[\varepsilon_{\nu'}(\kappa_x+q_x)-\varepsilon]\right)\\
&\qquad=2\frac{N_{\nu'}(\varepsilon)}{N_\nu(\varepsilon)}\sum_{\kappa_x}\sum_{\boldsymbol{q}}\Phi_{\nu\nu'}(\boldsymbol{q}_\perp,q_x)\delta(\varepsilon_\nu(\kappa_x)-\varepsilon)\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\delta[\varepsilon_{\nu'}(\kappa_x+q_x)-\varepsilon] \qquad \text{(A2)}
\end{aligned}$$

where

$$\Phi_{\nu\nu'}(\boldsymbol{q}_\perp,q_x)=w(q)G^2_{\nu\nu'}(\boldsymbol{q}_\perp)\overline{h}sq. \qquad \text{(A3)}$$

It can be proved by direct calculation that, if $\nu = \nu'$, then equation (A2) holds for an arbitrary even or odd function $\Phi_{\nu\nu'}(\boldsymbol{q}_\perp, q_x)$ of q_x. If $\nu \neq \nu'$, it does not hold and it depends on the particular form of this function. For the deformation acoustic potential in accordance with equation (2.5) $w(q) = w_0 q$ and $\Phi_{\nu\nu'}(\boldsymbol{q}_\perp, q_x) = w_0 \hbar s G^2_{\nu\nu'}(\boldsymbol{q}_\perp)(q_\perp^2 + q_x^2)$. Equation (A2) is realized for this function and we applied this in equation (A1).

For the third term in equation (4.4) we obtain

$$\begin{aligned}
\sum_{\kappa_x} \delta(\varepsilon_\nu(\kappa_x) - \varepsilon) \sum_{\boldsymbol{q}} w(q) G^2_{\nu\nu'}(\boldsymbol{q}_\perp) \frac{1}{2} (\hbar s q)^2 (2N_{\boldsymbol{q}} + 1) \{F_{0\nu'}[\varepsilon_{\nu'}(\kappa_x + q_x)] - F_{0\nu}(\varepsilon)\} \\
\times \frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2} \delta[\varepsilon_{\nu'}(\kappa_x + q_x) - \varepsilon] \\
= [F_{0\nu'}(\varepsilon) - F_{0\nu}(\varepsilon)] \sum_{\kappa_x} \sum_{\boldsymbol{q}} w(q) G^2_{\nu\nu'}(\boldsymbol{q}_\perp) \frac{1}{2} (\hbar s q)^2 (2N_{\boldsymbol{q}} + 1) \delta(\varepsilon_\nu(\kappa_x) - \varepsilon) \\
\times \frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2} \delta[\varepsilon_{\nu'}(\kappa_x + q_x) - \varepsilon] \\
+ \left[\frac{\mathrm{d}^2 F_{0\nu'}(\varepsilon)}{\mathrm{d}\varepsilon^2} \left(\sum_{\kappa_x} \sum_{\boldsymbol{q}} w(q) G^2_{\nu\nu'}(\boldsymbol{q}_\perp) \frac{1}{2} (\hbar s q)^2 (2N_{\boldsymbol{q}} + 1) \right.\right. \\
\left. \times \delta(\varepsilon_\nu(\kappa_x) - \varepsilon) \delta[\varepsilon_{\nu'}(\kappa_x + q_x) - \varepsilon] \right) \\
+ \frac{\mathrm{d}^2 F_{0\nu'}(\varepsilon)}{\mathrm{d}\varepsilon} \left(2 \sum_{\kappa_x} \sum_{\boldsymbol{q}} w(q) G^2_{\nu\nu'}(\boldsymbol{q}_\perp) \frac{1}{2} (\hbar s q)^2 (2N_{\boldsymbol{q}} + 1) \right. \\
\left.\left. \times \delta(\varepsilon_\nu(\kappa_x) - \varepsilon) \frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2} \delta[\varepsilon_{\nu'}(\kappa_x + q_x) - \varepsilon] \right) \right]. \qquad \text{(A4)}
\end{aligned}$$

For the equipartition approximation $N_{\boldsymbol{q}} \simeq T_0/(\hbar s q)$ and as a result the expression in large square brackets in equation (A4) has the same structure as the corresponding term in equation (A1). Then we can use equation (A1) to obtain the final expression for equation (A4) in the form

$$\begin{aligned}
[F_{0\nu'}(\varepsilon) - F_{0\nu}(\varepsilon)] T_0 \sum_{\kappa_x} \sum_{\boldsymbol{q}} w(q) G^2_{\nu\nu'}(\boldsymbol{q}_\perp) \hbar s q \delta(\varepsilon_\nu(\kappa_x) - \varepsilon) \frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2} \delta[\varepsilon_{\nu'}(\kappa_x + q_x) - \varepsilon] \\
+ T_0 \frac{N_\nu(\varepsilon)}{N_{\nu'}(\varepsilon)} \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \left(\frac{N_{\nu'}(\varepsilon)}{N_\nu(\varepsilon)} \sum_{\kappa_x} \sum_{\boldsymbol{q}} w(q) G^2_{\nu\nu'}(\boldsymbol{q}_\perp) \hbar s q \delta(\varepsilon_\nu(\kappa_x) - \varepsilon) \right. \\
\left. \times \delta[\varepsilon_{\nu'}(\kappa_x + q_x) - \varepsilon] \frac{\mathrm{d}F_{0\nu'}(\varepsilon)}{\mathrm{d}\varepsilon} \right). \qquad \text{(A5)}
\end{aligned}$$

By substituting equations (A1) and (A5) into equation (4.4) and using the definition of the density of states from equation (3.4), we have obtained equation (4.9) with the coefficients $A_{\nu\nu'}(\varepsilon)$ and $B_{\nu\nu'}(\varepsilon)$ which are given by equations (4.10) and (4.11), respectively.

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